Nilpotent Lie rings of order p^5

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1 Preliminaries

We need to consider four separate equivalence relations on 2×2 matrices over the field $\mathbb{Z}_p$ (p odd).

- $A \sim B$ if $B = PAP^{-1}$ for some non-singular P,
- $A \sim B$ if $B = \frac{1}{\det P} PAP^{-1}$ for some non-singular P,
- $A \sim B$ if $B = \lambda PAP^{-1}$ for some non-singular P, and some non-zero scalar λ,
- $A \sim B$ if $B = PAP^{-1}$ for some P with $\det P = 1$.

The first of these is similarity, which is well understood. First, two matrices can only be similar if they have the same characteristic polynomial (though this is not a sufficient condition for similarity). If A has characteristic polynomial $(x-\lambda)(x-\mu)$ $(\lambda \neq \mu)$ then A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ (and also to $\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}$). If A has characteristic polynomial $(x-\lambda)^2$ then A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. And if A has irreducible characteristic polynomial $x^2 - ax - b$ then A is similar to $\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix}$.

As we shall see, the answer is almost (but not quite!) the same for the other three equivalence relations. We call the fourth of these equivalence relations special-similarity. Note that if two matrices are special-similar, then they are also equivalent under the first three equivalence relations. First we need the following lemma.

Lemma 1 If $k \in \mathbb{Z}_p$ and $k \neq 0$ then every element of $\mathbb{Z}_p$ can be expressed in the form $a^2 + kb^2$ for some $a, b \in \mathbb{Z}_p$.

Proof. We assume that p is odd, since we are only interested in that case. (But the result is trivial for $p = 2$ anyway.) Recall that the multiplicative group of non-zero elements in $\mathbb{Z}_p$ is cyclic of even order. This implies that half the non-zero elements of $\mathbb{Z}_p$ are squares, and the other half are non-squares. Furthermore every non-square can be expressed in the form $a^2\omega$ for some øxed non-square ω. So if k is a non-square then there is no diŒculty: all squares in $\mathbb{Z}_p$ can be expressed in the form $a^2 + k0^2$ and all non-squares can be expressed in the form $0^2 + kb^2$.

So we assume that k is a square. The key step in the proof of Lemma 1 is to note that $a^2 + k$ must be a non-square for some a. For suppose that $a^2 + k$ is always a square. Since k is a square we have $2k = k + k$ is a square. But this implies that $3k$ is a square, and so on. But $\mathbb{Z}_p = \{k, 2k, 3k, \ldots\}$ so all elements of $\mathbb{Z}_p$ are squares, which is a contradiction.

Now suppose that a^2+k is a non-square. Then every non-square can be expressed in the form $(a^2 + k)b^2 = (ab)^2 + kb^2$. And every square can be expressed in the form $a^2 + k0^2$. □

Theorem 2 If A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ (possibly with $\lambda = \mu$) then A is special-similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. If A is similar to $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ then A is special similar to $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} \lambda & \omega \\ 0 & \lambda \end{pmatrix}$. Furthermore $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is not special-similar to $\begin{pmatrix} \lambda & \omega \\ 0 & \lambda \end{pmatrix}$. And if A is similar to $\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix}$ where $x^2 - ax - b$ is irreducible then A is special-similar to $\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix}$.

Proof. Suppose that A is similar to B. Then $B = PAP^{-1}$ for some non-singular P. Let $k = \det P$. Then we can write $P = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} Q$ for some matrix Q with $\det Q = 1$. So

$$B = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} QAQ^{-1} \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$QAQ^{-1} = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix} B \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.$$

So A is special-similar to $\begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix} B \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ for some k.

Now suppose that A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Then, by the above remarks, A is special similar to

$$\begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Next suppose that A is similar to $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Then A is special-similar to

$$\begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & \frac{1}{k} \\ 0 & \lambda \end{pmatrix}$$

for some $k \neq 0$. Now $\begin{pmatrix} \lambda & \frac{1}{k} \\ 0 & \lambda \end{pmatrix}$ is special-similar to

$$\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \lambda & \frac{1}{k} \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \lambda & \frac{a^2}{k} \\ 0 & \lambda \end{pmatrix}.$$

So A is special-similar to $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or to $\begin{pmatrix} \lambda & \omega \\ 0 & \lambda \end{pmatrix}$.

We need to prove that $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ and $\begin{pmatrix} \lambda & \omega \\ 0 & \lambda \end{pmatrix}$ are not special-similar. So suppose (for a contradiction) that

$$P \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} P^{-1} = \begin{pmatrix} \lambda & \omega \\ 0 & \lambda \end{pmatrix}$$

for some P of determinant 1. Write $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & \omega \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

And so

$$\begin{pmatrix} a\lambda & a + b\lambda \\ c\lambda & c + d\lambda \end{pmatrix} = \begin{pmatrix} a\lambda + \omega c & b\lambda + \omega d \\ c\lambda & d\lambda \end{pmatrix}.$$

It follows that $c = 0$ and $a = \omega d$. But then $\det P = \omega d^2 \neq 1$, which gives our contradiction.

Finally suppose that A has irreducible characteristic polynomial $x^2 - ax - b$, so that A is similar to $\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix}$. Then A is special-similar to

$$\begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{b}{k} \\ k & a \end{pmatrix}$$

for some $k \neq 0$. We show that $\begin{pmatrix} 0 & \frac{b}{k} \\ k & a \end{pmatrix}$ is special-similar to $\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix}$.

We need to ønd a matrix $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of determinant 1 such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & \frac{b}{k} \\ k & a \end{pmatrix} = \begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

This requires

$$\begin{pmatrix} \beta k & \frac{\alpha b+\beta a k}{k} \\ \delta k & \frac{\gamma b+\delta a k}{k} \end{pmatrix} = \begin{pmatrix} \gamma b & b\delta \\ \alpha + a\gamma & \beta + \delta a \end{pmatrix}.$$

So we need $\beta k = \gamma b$, $\delta k = \alpha + a\gamma$. So we can take

$$P = \begin{pmatrix} \delta k - a\gamma & \frac{\gamma b}{k} \\ \gamma & \delta \end{pmatrix}$$

for any γ, δ, provided $\det P = 1$. Now

$$\det \begin{pmatrix} \delta k - a\gamma & \frac{\gamma b}{k} \\ \gamma & \delta \end{pmatrix} = \frac{\delta^2 k^2 - \delta k a\gamma - \gamma^2 b}{k}.$$

So we want to choose γ, δ so that $\delta^2 k^2 - \delta k a \gamma - \gamma^2 b = k$. Now

$$\delta^2 k^2 - \delta k a\gamma - \gamma^2 b = (\delta k - \frac{a\gamma}{2})^2 - (b + \frac{a^2}{4})\gamma^2.$$

Furthermore $b + a^2/4 \neq 0$ since $x^2 - ax - b$ is irreducible. So, by Lemma 1, $\delta^2 k^2 - \delta k a\gamma - \gamma^2 b$ takes on all values in $\mathbb{Z}_p$ as γ, δ range over $\mathbb{Z}_p$. In particular, $\delta^2 k^2 - \delta k a \gamma - \gamma^2 b = k$ for some choice of γ, δ.

This completes the proof of Theorem 2. □

Lemma 3 If A has characteristic polynomial $x^2 - ax - b$, and if $B = (1/\det P)PAP^{-1}$ for some P then B has characteristic polynomial $x^2 - akx - bk^2$ for some $k \neq 0$. Furthermore, if A has characteristic polynomial $x^2 - ax - b$ and if $k \neq 0$, then there is a non-singular matrix P such that $(1/\det P)PAP^{-1}$ has characteristic polynomial $x^2 - akx - bk^2$.

Proof. Let $B = (1/\det P)PAP^{-1}$ and suppose that $\det P = 1/k$. Then $B = kPAP^{-1}$ and PAP^{-1} has characteristic polynomial $x^2 - ax - b$. So $B = kC$ for some matrix C with characteristic polynomial $x^2 - ax - b$ and an easy calculation shows that this implies that B has characteristic polynomial $x^2 - akx - bk^2$.

Conversely, let $k \neq 0$, and let P be any matrix with determinant $1/k$. Let

$$B = kPAP^{-1} = \frac{1}{\det P} PAP^{-1}.$$

Then the same calculation as above shows that the characteristic polynomial of B is $x^2 - akx - bk^2$. □

Lemma 4 If A is any 2×2 matrix, then there is a non-singular matrix P such that $(1/\det P)PAP^{-1}$ has characteristic polynomial $x^2 - x - c$ or x^2 or $x^2 - 1$ or $x^2 - \omega$. Furthermore if A and B are two matrices with characteristic polynomials of this form, but where the characteristic polynomials of A and B are dicerent, then

$$B \neq \frac{1}{\det P} PAP^{-1}$$

for any P.

Proof. Lemma 4 follows from Lemma 3. Lemma 3 implies that if A has characteristic polynomial $x^2 - ax - b$ where $a \neq 0$ then there is some non-singular matrix P such that $(1/\det P)PAP^{-1}$ has characteristic polynomial $x^2 - x - c$ (with $c = b/a^2$). It also implies $(1/\det P)PAP^{-1}$ cannot have characteristic polynomial $x^2 - x - d$ with $d \neq c$, or characteristic polynomial x^2 or $x^2 - 1$ or $x^2 - \omega$.

So consider a matrix A with characteristic polynomial $x^2 - b$. Then by Lemma 3, if P is any non-singular matirx then $(1/\det P)PAP^{-1}$ has characteristic polynomial $x^2 - bk^2$ for some $k \neq 0$. We can choose k so that $bk^2 = 0$ or 1 or ω (depending on whether $b = 0$, or b is a square, or b is a non-square). □

Theorem 5 If A and B are 2×2 matrices then we say that A is equivalent to B if $B = (1/\det P)PAP^{-1}$ for some non-singular matrix P. Every 2×2 matrix is equivalent to a matrix of one of the following forms:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},$$

or to a matrix of the form

$$\begin{pmatrix} 0 & c \\ 1 & 1 \end{pmatrix},$$

where $x^2 - x - c$ is irreducible. Furthermore none of these matrices are equivalent to each other, except that if $\lambda \neq 0$ then $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

Proof. Let A be a 2×2 matrix.

First we consider the case when the characteristic polynomial of A is irreducible. Let the characteristic polynomial be $x^2 - ax - b$ with $a \neq 0$. Then, by Lemma 4, we can find P so that $(1/\det P)PAP^{-1}$ has characteristic polynomial $x^2 - x - c$ (with $c = b/a^2$). This polynomial is also irreducible, and so (by Theorem 2) A is equivalent to $\begin{pmatrix} 0 & c \\ 1 & 1 \end{pmatrix}$.

Next suppose that the characteristic polynomial of A is irreducible and is of the form $x^2 - b$. Then b is not a square, and so by Lemma 4 we can find P so that $(1/\det P)PAP^{-1}$ has characteristic polynomial $x^2 - \omega$. By Theorem 2 this implies that A is equivalent to $\begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

Next suppose that the characteristic polynomial of A is $(x - \lambda)(x - \mu)$ with $\lambda \neq \mu$. By Theorem 2, there is a matrix P of determinant 1 with

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

So A is equivalent to

$$k \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k\lambda & 0 \\ 0 & k\mu \end{pmatrix}$$

for any $k \neq 0$. So A is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1}\mu \end{pmatrix}$. And if $\mu \neq 0$ then we also have $A \sim \begin{pmatrix} 1 & 0 \\ 0 & \lambda\mu^{-1} \end{pmatrix}$.

If the characteristic polynomial of A is x^2 then either $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, or (by Theorem 2) $A \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$.

Finally suppose that the characteristic polynomial of A is $(x-\lambda)^2$ for some $\lambda \neq 0$. Then (by Theorem 2) A is special-similar to $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}$ for some $\mu = 0$, 1 or ω. So

$$A \sim \frac{1}{\lambda} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.$$

We need to show that two matrices from the statement of Theorem 5 can only be equiavalent as speciøed in the statement of the theorem.

As we showed above, if the characteristic polynomial of A is irreducible then A is equivalent to a matrix with (irreducible) characteristic polynomial $x^2 - x - c$ or $x^2 - \omega$, and hence similar to $\begin{pmatrix} 0 & c \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$. It is clear from the proof of Lemma 3 that no two matrices with distinct characteristic polynomials of this form can be equivalent.

If A has characteristic polynomial $(x-\lambda)(x-\mu)$ with $\lambda \neq \mu$ and $\lambda \neq 0$ then, as we showed above, A is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1}\mu \end{pmatrix}$. This matrix has characteristic polynomial $(x-1)(x-\lambda^{-1}\mu)$. It is clear from the proof of Lemma 3 that any matrix equivalent to A has characteristic polynomial $(x-k)(x-\lambda^{-1}\mu k)$ for some $k \neq 0$. The only matrices in the statement of Theorem 5 with characteristic polynomial of this form are $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1}\mu \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \lambda\mu^{-1} \end{pmatrix}$ in the case when $\mu \neq 0$.

If A has characteristic polynomial $(x-\lambda)^2$ then, as above, A is equivalent to one of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix},$$

and it is straightforward to show that none of these are equivalent to each other. Furthermore it follows from Lemma 3 that any matrix equivalent to A has characteristic polynomial $(x-\mu)^2$ for some μ, and so A cannot be equivalent to any other of the matrices from the statement of Theorem 5. □

Theorem 6 If A and B are 2×2 matrices then we say that A is equivalent to B if $B = \lambda PAP^{-1}$ for some non-singular matrix P and some non-zero scalar λ. Every

2×2 matrix is equivalent ot a matrix of one of the following forms:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},$$

or to a matrix of the form

$$\begin{pmatrix} 0 & c \\ 1 & 1 \end{pmatrix},$$

where $x^2 - x - c$ is irreducible. Furthermore none of these matrices are equivalent to each other, except that if $\lambda \neq 0$ then $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

Proof. Clearly two matrices are equivalent under this equivalence relation if they are equivalent under the equivalence relation deøned in Theorem 5. So to show that every 2×2 matrix is equivalent (under this equivalence relation) to one of the above matrices, we need only show that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$$

and that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}.$$

This is immediate, since these pairs of matrices are similar. None of the matrices listed in the statement of Theorem 6 are equivalent to each other by an argument similar to the one given in the proof of Theorem 5. □

It is convenient to list the non-cyclic nilpotent Lie algebras of order less than or equal to p^4.

$$\langle a, b \,|\, \text{class } 1 \rangle \tag{2.1}$$

$$\langle a, b, c \,|\, \text{class } 1 \rangle \tag{3.1}$$

$$\langle a, b \,|\, pa,\ pb,\ \text{class } 2 \rangle \tag{3.2}$$

$$\langle a, b \,|\, pa - ba,\ pb,\ \text{class } 2 \rangle \tag{3.3}$$

$$\langle a, b \,|\, ba,\ pb,\ \text{class } 2 \rangle \tag{3.4}$$

$$\langle a, b, c, d \,|\, \text{class } 1 \rangle \tag{4.1}$$

$$\langle a, b, c \mid ba\,, ca,\ cb,\ pb,\ pc, \text{ class } 2\rangle \tag{4.2}$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb,\ pc, \text{ class } 2\rangle \tag{4.3}$$

$$\langle a, b, c \mid ca,\ cb,\ pa - ba,\ pb,\ pc, \text{ class } 2\rangle \tag{4.4}$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb,\ pc - ba, \text{ class } 2\rangle \tag{4.5}$$

$$\langle a, b \mid ba, \text{ class } 2\rangle \tag{4.6}$$

$$\langle a, b \mid pb, \text{ class } 2\rangle \tag{4.7}$$

$$\langle a, b \mid pb - ba, \text{ class } 2\rangle \tag{4.8}$$

$$\langle a, b \mid bab,\ pa,\ pb, \text{ class } 3\rangle \tag{4.9}$$

$$\langle a, b \mid bab,\ pa - baa,\ pb, \text{ class } 3\rangle \tag{4.10}$$

$$\langle a, b \mid bab,\ pa,\ pb - baa, \text{ class } 3\rangle \tag{4.11}$$

$$\langle a, b \mid bab,\ pa,\ pb - \omega baa, \text{ class } 3\rangle \tag{4.12}$$

$$\langle a, b \mid ba,\ \ pb, \text{ class } 3\rangle \tag{4.13}$$

$$\langle a, b \mid ba - p^2 a,\ \ pb, \text{ class } 3\rangle \tag{4.14}$$

2 Five generators

There is only one øve generator Lie ring of order p^5:

$$\langle a, b, c, d, e \mid \text{class } 1\rangle. \tag{5.1}$$

3 Four generators

If L is a four generator nilpotent Lie ring of order p^5 then L is an immediate descendant of 4.1. If L is abelian then we have

$$\langle a, b, c, d \mid ba, ca, da, cb, db, dc, pb, pc, pd, \text{ class } 2\rangle. \tag{5.2}$$

If L is not abelian then we may suppose that $pL \leq L^2$, and that L^2 is spanned by ba. We may also suppose that

$$ca = da = cb = db = 0$$

and that $dc = 0$ or ba. We may also assume that $\langle a, b\rangle$ is isomorphic to 3.2 or 3.3.

First suppose that $dc = 0$. If $pc = pd = 0$ then we have

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pa, pb, pc, pd, \text{ class } 2\rangle \tag{5.3}$$

or

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pa - ba, pb, pc, pd, \text{ class } 2\rangle. \tag{5.4}$$

On the other hand, if pc and pd are not both zero then we may assume that $pc = ba$ and that $pd = 0$. Replacing a by $a - c$ if necessary, we may suppose that $pa = 0$. So we have

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pa, pb, pc - ba, pd, \text{ class } 2\rangle. \tag{5.5}$$

Next suppose that $dc = ba$. We may assume that $\langle c, d\rangle$ is also isomorphic to 3.2 or 3.3. So we obtain

$$\langle a, b, c, d \mid ca, da, cb, db, dc - ba, pa, pb, pc, pd, \text{ class } 2\rangle, \tag{5.6}$$

$$\langle a, b, c, d \mid ca, da, cb, db, dc - ba, pa - ba, pb, pc, pd, \text{ class } 2\rangle, \tag{5.7}$$

and

$$\langle a, b, c, d \mid ca, da, cb, db, dc - ba, pa - ba, pb, pc - ba, pd, \text{ class } 2\rangle.$$

But if we let $a' = a + c$, $b' = b + d$, $c' = a - c$, $d' = b - d$ in this last algebra, then we see that it is isomorphic to 5.6.

Algebra 5.2 is the only abelian algebra among 5.2 ˘ 5.7. In 5.3 ˘ 5.5 the centre is spanned by c, d, ba, and in 5.6 and 5.7 the centre is spanned by ba. Algebras 5.3 and 5.6 have characteristic p, whereas the other algebras do not. And in 5.4 the centre has characteristic p, whereas in 5.5 the centre does not have characteristic p. So the algebras 5.2 ˘ 5.7 are all distinct.

4 Three generators

If L is a three generator nilpotent Lie ring of order p^5 then L is an immediate descendant of 3.1, or of one of 4.2 ˘ 4.5.

4.1 Descendants of 3.1

4.1.1 L^2 abelian

If L^2 is abelian then we have

$$\langle a, b, c \,|\, ba,\ ca,\ cb,\ pc,\ \text{class } 2\rangle. \tag{5.8}$$

4.1.2 L^2 has order p

If L^2 has order p then we may assume that L^2 is spanned by ba and that $ca = cb = 0$. If $pc = 0$ then $\langle a, b\rangle$ is isomorphic to 4.7 or 4.8, so that we have

$$\langle a, b, c \,|\, ca,\ cb,\ pb,\ pc,\ \text{class } 2\rangle, \tag{5.9}$$

$$\langle a, b, c \,|\, ca,\ cb,\ pb - ba,\ pc,\ \text{class } 2\rangle. \tag{5.10}$$

On the other hand, if $pc \neq 0$, then there are two cases to consider. In the ørst case $pc \in L^2$, and in the second case $pc \notin L^2$.

So consider the case when pc is non-zero, but $pc \in L^2$. We may assume that $pc = ba$. At least one of pa, pb must lie outside L^2, and we may suppose that $pa \notin L^2$. So pL is spanned by pa and pc, and replacing b by $b - \lambda a - \mu c$ for suitable λ, μ we may suppose that $pb = 0$. This gives

$$\langle a, b, c \,|\, ca,\ cb,\ pb,\ pc - ba,\ \text{class } 2\rangle. \tag{5.11}$$

Finally consider the case when $pc \notin L^2$. Replacing a and b by $a - \lambda c$, $b - \mu c$ for suitable λ, μ we may suppose that $pa, pb \in L^2$, so that $\langle a, b\rangle$ is isomorphic to 3.2 or 3.3. This gives

$$\langle a, b, c \,|\, ca,\ cb,\ pa,\ pb,\ \text{class } 2\rangle, \tag{5.12}$$

$$\langle a, b, c \,|\, ca,\ cb,\ pa - ba,\ pb,\ \text{class } 2\rangle. \tag{5.13}$$

In 5.9 and 5.10 the centre of L has order p^2, and in 5.11 ˘ 5.13 it has order p. In 5.9 and 5.12 pL has order p, but in 5.10, 5.11 and 5.13 it has order p^2. If C is the centre of L, then $pC \leq L^2$ in 5.11 but $pC \not\leq L^2$ in 5.13. So these 5 algebras are distinct.

4.1.3 L^2 has order p^2

If L^2 has order p^2 then we may assume that L^2 is spanned by ba, ca, and that $cb = 0$. Note that

$$B = \langle b, c\rangle + L^2 + pL$$

is a characteristic subalgebra of L of order p^4.

If $pB = \{0\}$ then we have $pb = pc = 0$ and we may assume that $pa = 0$ or ba. This gives

$$\langle a, b, c \,|\, cb,\ pa,\ pb,\ pc,\ \text{class } 2\rangle, \tag{5.14}$$

$$\langle a, b, c \,|\, cb,\ pa - ba,\ pb,\ pc,\ \text{class } 2 \rangle. \tag{5.15}$$

Note that 5.14 has characteristic p, whereas 5.15 does not.

If pB has order p then we may suppose that $pb \neq 0$ and that $pc = 0$. Let $pb = \beta ba + \gamma ca$. If $\beta = 0$ then we can scale a so that $\gamma = 1$. And if $\beta \neq 0$ we can scale a so that $\beta = 1$, and then replacing b by $b + \gamma c$ we have $pb = ba$. So we may assume that $pb = ba$ or ca, and that $pc = 0$. Let $pa = \lambda ba + \mu ca$. If $pb = ba$ then replacing a by $a - \lambda b$ we have $pa = \mu ca$, and scaling c we can take $\mu = 0$ or 1. And if $pb = ca$ then replacing a by $a - \mu b$ we have $pa = \lambda ba$, and by scaling b and c we can take $\lambda = 0$ or 1. This gives

$$\langle a, b, c \,|\, cb,\ pa,\ pb - ba,\ pc,\ \text{class } 2 \rangle, \tag{5.16}$$

$$\langle a, b, c \,|\, cb,\ pa - ca,\ pb - ba,\ pc,\ \text{class } 2 \rangle, \tag{5.17}$$

$$\langle a, b, c \,|\, cb,\ pa,\ pb - ca,\ pc,\ \text{class } 2 \rangle, \tag{5.18}$$

$$\langle a, b, c \,|\, cb,\ pa - ba,\ pb - ca,\ pc,\ \text{class } 2 \rangle. \tag{5.19}$$

In 5.16 and 5.18 pL has order p, whereas in 5.17 and 5.19 pL has order p^2. If we let

$$C = \langle c \rangle + L^2 + pL$$

then C is the unique subalgebra of B of order p^3 and characteristic p. So C is a characteristic subalgebra of L. In 5.18 and 5.19 $pB \leq CL$, but in 5.16 and 5.17 $pB \not\leq CL$. So these four algebras are distinct.

Finally suppose that pB has order p^2. Then pb and pc span $L^2 = pL$, and replacing a by $a - \lambda b - \mu c$ for suitable λ, μ we may suppose that $pa = 0$. We can write

$$\begin{pmatrix} pb \\ pc \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}$$

for some 2×2 matrix A, and then L is determined by the entries in A, which we may view as elements in the field $\mathbb{Z}_p$. If we let

$$\begin{pmatrix} b' \\ c' \end{pmatrix} = P \begin{pmatrix} b \\ c \end{pmatrix}$$

for some non-singular 2×2 matrix P with entries in $\mathbb{Z}_p$ then

$$\begin{pmatrix} pb' \\ pc' \end{pmatrix} = PAP^{-1} \begin{pmatrix} b'a \\ c'a \end{pmatrix}.$$

So A and PAP^{-1} give isomorphic algebras. Also, scaling a transforms A to λA for some non-zero λ. So A and λPAP^{-1} give isomorphic algebras. Conversely the argument above shows that if A and B give isomorphic algebras then $B = \lambda PAP^{-1}$

for some non-singular P and some non-zero λ. Also, we are assuming that A is non-singular, so by Theorem 6 we can take A to be one of the folowing matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} (\lambda \neq 0), \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ 1 & 1 \end{pmatrix},$$

where $x^2 - x - \alpha$ is irreducible. Thus we obtain $(p+1)/2$ algebras

$$\langle a, b, c \mid cb,\ pa,\ pb - ba,\ pc - \lambda ca, \text{ class } 2\rangle, \tag{5.20}$$

with $\lambda \neq 0$, where λ and λ^{-1} give isomorphic algegbras;

$$\langle a, b, c \mid cb,\ pa,\ pb - ba - ca,\ pc - ca, \text{ class } 2\rangle; \tag{5.21}$$

$$\langle a, b, c \mid cb,\ pa,\ pb - \omega ca,\ pc - ba, \text{ class } 2\rangle; \tag{5.22}$$

and $(p-1)/2$ algebras

$$\langle a, b, c \mid cb,\ pa,\ pb - \alpha ca,\ pc - ba - ca, \text{ class } 2\rangle, \tag{5.23}$$

where $1 + 4\alpha$ is not a square.

4.2 Descendants of 4.2

If L is an immediate descendant of 4.2 of order p^5 then L is generated by a, b, c. The subalgebra L_2 is spanned by pa modulo L_3, and L_3 is spanned by p^2a. The elements ba, ca, cb, pb, pc are all linear multiples of p^2a. Replacing b by $b - \lambda pa$ for suitable λ we may suppose that $pb = 0$. Similarly we may suppose that $pc = 0$.

If L is abelian then we have

$$\langle a, b, c \mid ba,\ ca,\ cb,\ pb,\ pc, \text{ class } 3\rangle. \tag{5.24}$$

So suppose that L is not abelian. The subalgebra $\langle pa, b, c\rangle$ is characteristic, and its derived algebra is spanned by cb. If $cb \neq 0$ then we may suppose that $cb = p^2a$, and replacing a by $a + \lambda b + \mu c$ for suitable λ, μ we may suppose that $ba = ca = 0$. This gives

$$\langle a, b, c \mid ba,\ ca,\ cb - p^2a,\ pb,\ pc, \text{ class } 3\rangle. \tag{5.25}$$

On the other hand, if $cb = 0$ and one of ba, ca is non-zero, then we may suppose that $ba = p^2a$, and replacing c by $c - \nu b$ foru suitable ν we may suppose that $ca = 0$. This gives

$$\langle a, b, c \mid ba - p^2a,\ ca,\ cb,\ pb,\ pc, \text{ class } 3\rangle. \tag{5.26}$$

4.3 Descendants of 4.3

If L is an immediate descendant of 4.3 of order p^5 then L is generated by a, b, c. The subalgebra L_2 is spanned by ba modulo L_3, and L_3 is spanned by baa, bab. The elements ca, cb, pa, pb, pc all lie in L_3. We may suppose that L_3 is spanned by baa and that $bab = 0$. Replacing c by $c - \lambda ba$ for suitable λ we may suppose that $ca = 0$. Scaling c we may suppose that $cb = 0$ or baa.

First suppose that $cb = 0$. The subalgebra $\langle a, b\rangle$ is isomorphic to one of 4.9 ˘ 4.12, so if $pc = 0$ we have

$$\langle a, b, c \mid ca,\ cb,\ bab,\ pa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{5.27}$$

$$\langle a, b, c \mid ca,\ cb,\ bab,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{5.28}$$

$$\langle a, b, c \mid ca,\ cb,\ bab,\ pa,\ pb - baa,\ pc,\ \text{class } 3\rangle, \tag{5.29}$$

$$\langle a, b, c \mid ca,\ cb,\ bab,\ pa,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle. \tag{5.30}$$

But if $pc \neq 0$ then replacing a by $a - \mu c$ for suitable μ we may suppose that $pa = 0$. Similarly we may suppose that $pb = 0$. Scaling c we may suppose that $pc = baa$. This gives

$$\langle a, b, c \mid ca,\ cb,\ bab,\ pa,\ pb,\ pc - baa,\ \text{class } 3\rangle. \tag{5.31}$$

Next suppose that $cb = baa$. Once again, $\langle a, b\rangle$ is isomorphic to one of 4.9 ˘ 4.12, so if $pc = 0$ we have

$$\langle a, b, c \mid ca,\ cb - baa,\ bab,\ pa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{5.32}$$

$$\langle a, b, c \mid ca,\ cb - baa,\ bab,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{5.33}$$

$$\langle a, b, c \mid ca,\ cb - baa,\ bab,\ pa,\ pb - baa,\ pc,\ \text{class } 3\rangle, \tag{5.34}$$

$$\langle a, b, c \mid ca,\ cb - baa,\ bab,\ pa,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle. \tag{5.35}$$

And if $pc \neq 0$ then scaling b we may take $pc = baa$ and (as above) we obtain

$$\langle a, b, c \mid ca,\ cb - baa,\ bab,\ pa,\ pb,\ pc - baa,\ \text{class } 3\rangle. \tag{5.36}$$

Note that in all the algebras 5.27 ˘ 5.36 the subalgebra $C = \langle c, ba, baa\rangle$ is characteristic. Note that $pC = \{0\}$ in all these algebras except for 5.30 and 5.36. The centralizer of L^2 is

$$B = \langle b, c, ba, baa\rangle,$$

so B is also characteristic. In 5.27 ˘ 5.31 $CB = \{0\}$, and in 5.32 ˘ 5.36 $CB \neq \{0\}$. So it remains to distinguish the algebras 5.27 ˘ 5.30 from each other, and to distinguish the algebras 5.32 ˘ 5.35 from each other. However if a', b' are any elements of L which span L modulo C in 5.27 ˘ 5.30, or in 5.32 ˘ 5.35, then $\langle a', b'\rangle$ is isomorphic to $\langle a, b\rangle$, so these algebras are all distinct.

4.4 Descendants of 4.4

Algebra 4.4 is terminal.

4.5 Descendants of 4.5

Algebra 4.5 is also terminal.

5 Two generators

If L is a two generator nilpotent Lie ring of order p^5, then L is an immediate desendant of one of 2.1, 3.2 ˘ 3.4, or 4.6 ˘ 4.14.

5.1 Descendants of 2.1

The only immediate descendant of 2.1 of order p^5 is

$$\langle a, b \,|\, \text{class } 2\rangle. \tag{5.37}$$

5.2 Descendants of 3.2

If L is an immediate descdant of 3.2 of order p^5 then L is generated by a, b, L^2 is spanned by ba modulo L^3, and L^3 has order p^2 and is spanned by baa, bab. We also have $pa, pb \in L^3$.

If $pa = pb = 0$ then we have

$$\langle a, b \,|\, pa,\ pb,\ \text{class } 3\rangle. \tag{5.38}$$

If pa and pb span a subalgebra of order p then we may suppose that $pa = \alpha baa + \beta bab$, $pb = 0$. If $\alpha = 0$ then scaling b we may suppose that $\beta = 1$ or ω. And if $\alpha \neq 0$ then scaling b we may suppose that $\alpha = 1$, and replacing a by $a + \beta b$ we may suppose that $pa = baa$. So we have

$$\langle a, b \,|\, pa - bab,\ pb,\ \text{class } 3\rangle, \tag{5.39}$$

$$\langle a, b \,|\, pa - \omega bab,\ pb,\ \text{class } 3\rangle, \tag{5.40}$$

$$\langle a, b \,|\, pa - baa,\ pb,\ \text{class } 3\rangle. \tag{5.41}$$

Clearly these three algebras are distinct.

Finally suppose that pa and pb span a subalgebra of order p^2. We can write

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

where A is a non-singular 2×2 matrix with entries in $\mathbb{Z}_p$. If we let

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = P \begin{pmatrix} a \\ b \end{pmatrix}$$

for some non-singular 2×2 matrix with entries in $\mathbb{Z}_p$ then

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \frac{1}{\det P} PAP^{-1} \begin{pmatrix} b'a'a' \\ b'a'b' \end{pmatrix}.$$

So two matrices A_1 and A_2 give isomorphic algebras if and only if

$$A_2 = \frac{1}{\det P} PA_1P^{-1}$$

for some non-singular matrix P. So, by Theorem 5, we have one of the following algebras

$$\langle a, b \,|\, pa - baa,\ pb - \lambda bab,\ \text{class } 3\rangle, \tag{5.42}$$

with $\lambda \neq 0$ and λ, λ^{-1} giving isomorphic algebras ($(p+1)/2$ algebras);

$$\langle a, b \,|\, pa - baa - bab,\ pb - bab,\ \text{class } 3\rangle, \tag{5.43}$$

$$\langle a, b \,|\, pa - baa - \omega bab,\ pb - bab,\ \text{class } 3\rangle, \tag{5.44}$$

$$\langle a, b \,|\, pa - \omega bab,\ pb - baa,\ \text{class } 3\rangle; \tag{5.45}$$

$$\langle a, b \,|\, pa - \alpha bab,\ pb - baa - bab,\ \text{class } 3\rangle, \tag{5.46}$$

where $1 + 4\alpha$ is not a square ($(p-1)/2$ algebras).

5.3 Descendants of 3.3

Algebra 3.3 is terminal.

5.4 Descendants of 3.4

Algebra 3.4 has no immediate descendants of order p^5.

5.5 Descendants of 4.6

Let L be an immediate descendant of 4.6 of order p^5. Then L is generated by a, b, L_2 is generated modulo L_3 by pa, pb, and L_3 has order p and is generated by p^2a, p^2b. Also $ba \in L_3$. Clearly we may assume that L_3 is generated by p^2a, and that $p^2b = 0$. Scaling, we may assume that $ba = 0$ or p^2a. So we have

$$\langle a, b \,|\, ba,\ p^2b,\ \text{class } 3\rangle, \tag{5.47}$$

$$\langle a, b \,|\, ba - p^2a,\ p^2b,\ \text{class } 3\rangle. \tag{5.48}$$

5.6 Descendants of 4.7

Let L be an immediate descendant of 4.7 of order p^5. Then L is generated by a, b, L_2 is generated by ba, pa modulo L_3, and L_3 has order p and is generated by baa, bab, p^2a. Furthermore $pb \in L_3$.

First suppose that $baa = bab = 0$, so that L_3 is generated by p^2a and $pb = \alpha p^2 a$ for some α. Then replacing b by $b - \alpha pa$ we have $pb = 0$, which gives

$$\langle a, b \mid baa,\ bab,\ pb,\ \text{class } 3\rangle. \tag{5.49}$$

Next, assume that $baa \neq 0$, $bab = 0$. Then p^2a and pb are scalar multiples of baa. Scaling, we may assume that $p^2a = 0$ or baa. If $p^2a = 0$ then, scaling, we may assume that $pb = 0$, baa, or ωbaa. And if $p^2a = baa$ then (as above) we may assume that $pb = 0$. So we have

$$\langle a, b \mid bab,\ p^2a,\ pb,\ \text{class } 3\rangle, \tag{5.50}$$

$$\langle a, b \mid bab,\ p^2a,\ pb - baa,\ \text{class } 3\rangle, \tag{5.51}$$

$$\langle a, b \mid bab,\ p^2a,\ pb - \omega baa,\ \text{class } 3\rangle, \tag{5.52}$$

$$\langle a, b \mid bab,\ p^2a - baa,\ pb,\ \text{class } 3\rangle. \tag{5.53}$$

Finally, assume that $bab \neq 0$. Then, replacing a by $a - \lambda b$ for suitable λ we may assume that $baa = 0$. We also may assume that p^2a and pb are scalar multiples of bab. Scaling, we may assume that $p^2a = 0$, bab, or ωbab. If $p^2a \neq 0$ then we may assume that $pb = 0$, and if $p^2a = 0$ then we may assume that $pb = 0$ or bab. So we have

$$\langle a, b \mid baa,\ p^2a,\ pb,\ \text{class } 3\rangle, \tag{5.54}$$

$$\langle a, b \mid baa,\ p^2a,\ pb - bab,\ \text{class } 3\rangle, \tag{5.55}$$

$$\langle a, b \mid baa,\ p^2a - bab,\ pb,\ \text{class } 3\rangle, \tag{5.56}$$

$$\langle a, b \mid baa,\ p^2a - \omega bab,\ pb,\ \text{class } 3\rangle. \tag{5.57}$$

5.7 Descendants of 4.8

Let L be an immediate descendant of 4.8 of order p^5. Then L is generated by a, b, L_2 is generated by ba, pa modulo L_3, and L_3 has order p and is generated by baa, p^2a. Furthermore $bab = 0$ and $pb = ba$ modulo L_3.

First suppose that $baa = 0$. Then L_3 is generated by p^2a, and replacing b by $b - \lambda pa$ for suitable λ we may assume that $pb = ba$. So we have

$$\langle a, b \mid baa,\ pb - ba,\ \text{class } 3\rangle. \tag{5.58}$$

On the other hand, if $baa \neq 0$, then $p^2a = \alpha baa$ for some α. If we let $a' = a - \alpha b$, $b' = b$ then

$$\begin{aligned} b'a' &= ba, \\ pa' &= pa - \alpha pb = pa - \alpha ba \text{ modulo } L_3, \\ pb' &= ba \text{ modulo } L_3, \\ b'a'a' &= baa, \\ p^2a' &= p^2a - \alpha baa = 0. \end{aligned}$$

So we may assume that $p^2a = 0$. Now let

$$pb = ba + \beta baa,$$

and let $a' = a + \beta pa$. Then

$$\begin{aligned} ba' &= ba + \beta pba = ba + \beta baa, \\ pa' &= pa, \\ ba'a' &= baa, \\ p^2a' &= 0. \end{aligned}$$

So we have

$$\langle a, b \mid p^2a,\ pb - ba,\ \text{class } 3\rangle. \tag{5.59}$$

5.8 Descendants of 4.9

Let L be an immediate descendant of 4.9 of order p^5. Then L is generated by a, b, L_2 is generated by ba modulo L_3, L_3 is generated by baa modulo L_4, and L_4 is generated by $baaa$. Furthermore $bab, pa, pb \in L_4$.

Scaling, we may assume that $bab = 0$ or $baaa$. If $pb \neq 0$ then, then replacing a by $a - \lambda b$ for suitable λ we may assume that $pa = 0$. So we either have $pa = \alpha baaa$, $pb = 0$ for some α, or we have $pa = 0$, $pb = \beta baaa$ for some β.

If $bab = 0$, $pa = \alpha baaa$, $pb = 0$ then scaling b we may assume that $\alpha = 0$ or 1. This gives

$$\langle a, b \mid bab,\ pa,\ pb,\ \text{class } 4\rangle, \tag{5.60}$$

$$\langle a, b \mid bab,\ pa - baaa,\ pb,\ \text{class } 4\rangle. \tag{5.61}$$

If $bab = 0$, $pa = 0$, $pb = \beta baaa$. If $p = 2 \bmod 3$ then scaling a we may assume that $\beta = 0$ or 1, and if $p = 1 \bmod 3$ then we may assume that $\beta = 0, 1, \omega$ or ω^2. However the case $\beta = 0$ gives 5.60 again, so when $p = 2 \bmod 3$ we have

$$\langle a, b \mid bab,\ pa,\ pb - baaa,\ \text{class } 4\rangle, \tag{5.62}$$

and when $p = 1 \bmod 3$ then in addition to 5.62 we have

$$\langle a, b \mid bab,\ pa,\ pb - \omega baaa,\ \text{class } 4\rangle, \tag{5.63}$$

$$\langle a, b \mid bab,\ pa,\ pb - \omega^2 baaa,\ \text{class } 4\rangle. \tag{5.64}$$

Next, if $bab = baaa$, $pa = \alpha baaa$, $pb = 0$ then letting $a' = \lambda a$, $b' = \lambda^2 b$ we have

$$\begin{aligned} b'a'b' &= b'a'a'a', \\ pa' &= \lambda\alpha baaa = \lambda^{-4}\alpha b'a'a'a'. \end{aligned}$$

If $p = 3 \bmod 4$ then we can choose λ so that $\lambda^{-4}\alpha = 0, 1$, or ω, and if $p = 1 \bmod 4$ then we can choose λ so that $\lambda^{-4}\alpha = 0, 1, \omega, \omega^2$ or ω^3. So if $p = 3 \bmod 4$ we have

$$\langle a, b \mid bab - baaa,\ pa,\ pb,\ \text{class } 4\rangle, \tag{5.65}$$

$$\langle a, b \mid bab - baaa,\ pa - baaa,\ pb,\ \text{class } 4\rangle, \tag{5.66}$$

$$\langle a, b \mid bab - baaa,\ pa - \omega baaa,\ pb,\ \text{class } 4\rangle. \tag{5.67}$$

And if $p = 1 \bmod 4$ then, in addition to 5.65 ˘ 5.67 we have

$$\langle a, b \mid bab - baaa,\ pa - \omega^2 baaa,\ pb,\ \text{class } 4\rangle, \tag{5.68}$$

$$\langle a, b \mid bab - baaa,\ pa - \omega^3 baaa,\ pb,\ \text{class } 4\rangle. \tag{5.69}$$

Finally, if $bab = baaa$, $pa = 0$, $pb = \beta baaa$ then letting $a' = \lambda a$, $b' = \lambda^2 b$ we have

$$\begin{aligned} b'a'b' &= b'a'a'a', \\ pb' &= \lambda^2\beta baaa = \lambda^{-3}\beta b'a'a'a'. \end{aligned}$$

So if $p = 2 \bmod 3$ then we may assume that $\beta = 0$ or 1, and if $p = 1 \bmod 3$ then we may assume that $\beta = 0, 1, \omega$ or ω^2. The case $\beta = 0$ gives us 5.64 again, so if $p = 2 \bmod 3$ we have

$$\langle a, b \mid bab - baaa,\ pa,\ pb - baaa,\ \text{class } 4\rangle, \tag{5.70}$$

and if $p = 1 \bmod 3$ then in addition to 5.70 we have

$$\langle a, b \mid bab - baaa,\ pa,\ pb - \omega baaa,\ \text{class } 4\rangle, \tag{5.71}$$

$$\langle a, b \mid bab - baaa,\ pa,\ pb - \omega^2 baaa,\ \text{class } 4\rangle. \tag{5.72}$$

5.9 Descendants of 4.10

Algebra 4.10 is terminal.

5.10 Descendants of 4.11 and 4.12

Algebras 4.11 and 4.12 are also terminal.

5.11 Descendants of 4.13

If L is an immediate descendant of 4.13 of order p^5 then L is generated by a, b. Furthermore, L has p-class 4, and L_4 has order p and is generated by p^3a. We have $ba, pb \in L_4$. Replacing b by $b - \lambda p^2 a$ for suitable λ we may suppose that $pb = 0$. And scaling b we may suppose that $ba = 0$ or $p^3 a$. So we have

$$\langle a, b \mid ba,\ pb,\ \text{class } 4\rangle, \tag{5.73}$$

$$\langle a, b \mid ba - p^3 a,\ pb,\ \text{class } 4\rangle. \tag{5.74}$$

5.12 Descendants of 4.14

Algebra 4.14 is terminal.