The groups of order p^7

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Groups of order p^k for $k = 1, 2, \ldots, 6$

	$p = 2$	$p = 3$	$p \geq 5$
p	1	1	1
p^2	2	2	2
p^3	5	5	5
p^4	14	15	15
p^5	51	67	u
p^6	267	504	v

$$u = 2p + 61 + 2\gcd(p-1,3) + \gcd(p-1,4)$$

$$v = 3p^2 + 39p + 344 + 24\gcd(p-1,3) + 11\gcd(p-1,4) + 2\gcd(p-1,5)$$

Order p^7

$p=2$	$p=3$	$p=5$
2328	9310	34297

For $p > 5$ the number of groups of order p^7 is

$$
\begin{aligned}
&3p^5 + 12p^4 + 44p^3 + 170p^2 + 707p + 2455\\
&+(4p^2 + 44p + 291)\gcd(p-1,3)\\
&+(p^2 + 19p + 135)\gcd(p-1,4)\\
&+(3p + 31)\gcd(p-1,5)\\
&+4\gcd(p-1,7) + 5\gcd(p-1,8)\\
&+\gcd(p-1,9)
\end{aligned}
$$

Baker-Campbell-Hausdorff Formula

$\mathrm{e}^x.\mathrm{e}^y = \mathrm{e}^u$ where

$$\begin{aligned} u &= x + y - \frac{1}{2}[y,x] + \frac{1}{12}[y,x,x] - \frac{1}{12}[y,x,y] + \frac{1}{24}[y,x,x,y] \\ &\quad - \frac{1}{720}[y,x,x,x,x] - \frac{1}{180}[y,x,x,x,y] + \frac{1}{180}[y,x,x,y,y] \\ &\quad + \frac{1}{720}[y,x,y,y,y] - \frac{1}{120}[y,x,x,[y,x]] - \frac{1}{360}[y,x,y,[y,x]] + \ldots \end{aligned}$$

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$$

$[e^y, e^x] = e^w$ where

$$
\begin{aligned}
w &= [y,x] + \frac{1}{2}[y,x,x] + \frac{1}{2}[y,x,y] \\
&\quad + \frac{1}{6}[y,x,x,x] + \frac{1}{4}[y,x,x,y] + \frac{1}{6}[y,x,y,y] + \ldots
\end{aligned}
$$

If L is a Lie algebra define a group operation $\circ$ on L by setting

$$a \circ b = a + b - \frac{1}{2}[b, a] + \frac{1}{12}[b, a, a] - \frac{1}{12}[b, a, b] + \ldots$$

This works if L is a nilpotent Lie algebra over $\mathbb{Q}$, or if L is a Lie ring of order p^k and L is nilpotent of class at most $p - 1$.

If G is a group under $\circ$ and if $a, b \in G$ define

$$a + b = a \circ b \circ [b, a]_G^{\frac{1}{2}} \circ [b, a, a]_G^{-\frac{1}{12}} \circ [b, a, b]_G^{\frac{1}{12}} \circ \ldots$$

$$[b, a]_L = [b, a]_G \circ [b, a, a]_G^{-\frac{1}{2}} \circ [b, a, b]_G^{-\frac{1}{2}} \circ \ldots$$

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We need G to be nilpotent, and we need unique extraction of roots. So this works if G is a nilpotent torsion free divisible group, or if G is a finite p-group of class at most $p - 1$.

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This gives the Mal'cev correspondence between nilpotent Lie algebras over $\mathbb{Q}$ and nilpotent torsion free divisible groups. It also gives the Lazard correspondence between nilpotent Lie rings of order p^k and class at most $p-1$ and finite groups of order p^k and class at most $p-1$.

Classify groups of order p^7 for $p > 5$ by classifying nilpotent Lie rings of order p^7.

Use the Lie ring generation algorithm to classify the Lie rings. (Analogous to the p-group generation algorithm.)

Then use the Baker-Campbell-Hausdorff formula to translate Lie ring presentations into group presentations.

Lower exponent-p-central series

$$\begin{aligned} L_1 &= L \\ L_2 &= pL + [L, L] \\ L_3 &= pL_2 + [L_2, L] \\ &\dots \\ L_{n+1} &= pL_n + [L_n, L] \end{aligned}$$

a, b
$ba,\ pa,\ pb$
$baa,\ bab,\ pba,\ p^2a,\ p^2b$
...

L has p-class c if $L_{c+1} = \{0\}$, $L_c \neq \{0\}$.

Classify the nilpotent Lie rings of order p^k according to p-class.

If L has p-class $c > 1$ then we say that L is an immediate descendant of L/L_c.

To classify nilpotent Lie rings of order p^k, first classify all nilpotent Lie rings of order p^m for $m < k$.

If L has order p^m ($m < k$) find all immediate descendants of L of order p^k.

The p-covering ring

Let M be a nilpotent d-generator Lie ring of order p^m

The p-covering ring $\widehat{M}$ is the largest d-generator Lie ring with an ideal Z satisfying

- $Z \leq \zeta(\widehat{M})$
- $pZ = \{0\}$
- $\widehat{M}/Z \cong M$

Immediate descendants

If M has p-class c then every immediate descendant of M is of the form $\widehat{M}/T$ for some $T < Z$ such that

$$T + \widehat{M}_{c+1} = Z$$

If α is an automorphism of M then α lifts to an automorphism α^* of $\widehat{M}$.

$$\widehat{M}/S \cong \widehat{M}/T$$

if and only if $T = S\alpha^*$ for some α.

An example

$$\langle a, b \mid pa - baa - xbabb,\ pb - babb,\ \text{class} = 4 \rangle$$

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$a_1 = a,\ a_2 = b$
$a_3 = ba$
$a_4 = baa,\ a_5 = bab$
$a_6 = babb$

Computing the automorphism group

Consider an automorphism given by

$$\begin{aligned}
a_1 &\mapsto x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4 + x_5a_5 + x_6a_6 \\
a_2 &\mapsto x_7a_1 + x_8a_2 + x_9a_3 + x_{10}a_4 + x_{11}a_5 + x_{12}a_6
\end{aligned}$$

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The program gives the following conditions on $x_1, x_2, \ldots, x_{12}$ class by class.

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\end{aligned}
$$

At class 2, nothing.

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At class 3:

$$\begin{aligned} -x_1^2x_8 + x_1x_2x_7 + x_1 &= 0 \\ -x_1x_2x_8 + x_2^2x_7 &= 0 \\ x_7 &= 0 \end{aligned}$$

This gives $x_2 = x_7 = 0$, $x_8 = x_1^{-1}$.

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Set $x_2 = x_7 = 0$, and then at class 4 we have

$$\begin{aligned} -x_1^2x_8 + x_1 &= 0 \\ -xx_1x_8^3 + xx_1 &= 0 \\ -x_1x_8^3 + x_8 &= 0 \end{aligned}$$

These relations give $x_1 = x_8 = 1$.

The p-covering ring, $\widehat{L}$, has order p^9 with

$$\begin{aligned} a_7 &= babba \\ a_8 &= pa - baa - xbabb \\ a_9 &= pb - babb \end{aligned}$$

$\widehat{L}_5$ is generated by $a_7 = babba$, and so the immediate descendants of L are

$$\langle a, b \,|\, pa - baa - xbabb - ybabba,\; pb - babb - zbabba \rangle$$

with class 5 and $0 \leq y, z < p$.

If we apply the automorphism

$$\begin{aligned} a_1 &\mapsto a_1 + x_3a_3 + x_4a_4 + x_5a_5 + x_6a_6 \\ a_2 &\mapsto a_2 + x_9a_3 + x_{10}a_4 + x_{11}a_5 + x_{12}a_6 \end{aligned}$$

to $\widehat{L}$, then

$$\begin{aligned} babba &\mapsto babba \\ pa - baa - xbabb &\mapsto pa - baa - xbabb + (x_3^2 + 2x_5)babba \\ pb - babb &\mapsto pb - babb \end{aligned}$$

So we can take $y = 0$, and we have p non-isomorphic descendants for each value of x.

$$\langle a, b \,|\, pa - baa - xbabb,\ pb - babb - zbabba, \mathrm{class} = 5\rangle$$

Apply the Baker-Campbell-Hausdorff formula, and obtain the group relations

$$
\begin{aligned}
a^p &= [b,a,a] \cdot [b,a,b,b]^x \cdot [b,a,b,b,a]^{(x+1/3)} \\
b^p &= [b,a,b,b] \cdot [b,a,b,b,a]^z
\end{aligned}
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- IsIsomorphic(P,Q)
- StandardPresentation(P)
- IsIdenticalPresentation(P,Q)

```
p:=2;
while p lt 20 do
   for x in [0..p-1] do
      G:=Group<a,b|a^p=(b,a,a)*(b,a,b,b)^x,b^p=(b,a,b,b)>;
      P:=pQuotient(G,p,4);
      D:=Descendants(P:StepSizes:=[1]);
      print "p =",p," x =",x," ", Order(P) eq p^6, #D eq p;
   end for;
   if p eq 5 then readi i; end if;
   p:=NextPrime(p);
end while;
```