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ON THE SUPEREXTENSION OF THE KADOMTSEV-PETVIASHVILI EQUATION
П. И. Голод, С. З. Пакуляк

О суперрасширениях уравнения Кадомцева--Петвиашвили

В работе предложено новое суперобщение уравнения Кадомцева--Петвиашвили, из которого после редукции следует бигамильтоново суперуравнение Кортевега-де Фриза. Построена серия инволютивных законов сохранения и выписана иерархия высших уравнений для предлагаемого суперуравнения КП.

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On the Superextension of the Kadomtsev-Petviashvili Equation

A new supergeneralization of Kadomtsev-Petviashvili equation is proposed. Bi-Hamiltonian Korteweg-de Vries superequation follows from this super-KP equation after reduction. The set of involutory integrals is constructed and the hierarchy of higher equations for the proposed super-KP equation is written.

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О суперрасширениях уравнения Кадомцева--Петвиашвили

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Introduction

It was shown in Refs. [1,2], that there exist two ways for the nonlinear integrable Korteweg-de Vries equation to be supergeneralized. One of these equations [1] proves to be, in fact, supersymmetric (involutive integrals of motion for this equation commute with the generator of global supertransformation), and another one [4] is bi-Hamiltonic system, i.e., the integrals of motion, which are in involution, satisfy the relations [3]

\[ \Omega_{1} (S H_{n+\ell}) = \Omega_{2} (S H_{n}), \]

where \( \Omega_{1} \) and \( \Omega_{2} \) are the first and the second Hamiltonian structures for s-KdV superequation.

It is known that the supersymmetrical equation s-KdV [5,6]

\[ \begin{align*}
\left\{ \begin{array}{l}
\frac{d u}{d t} = \frac{1}{4} \frac{d u}{d x} + \frac{3}{2} \frac{d u}{d x} - \frac{3}{8} \frac{d^{2} u}{d x^{2}}; \\
\frac{d e}{d t} = \frac{1}{4} \frac{d e}{d x} + \frac{3}{8} \frac{d^{2} e}{d x^{2}} + \frac{3}{8} \frac{d (\Delta u)}{d x},
\end{array} \right.
\end{align*} \]

(2)

can be received by the reduction from the supersymmetric Kadomtsev-Petviashvili equations hierarchy, proposed in Ref. [6].

This hierarchy was constructed by means of the odd pseudodifferential operator \( \Lambda = D + \sum_{i=0}^{\infty} d_{i} D^{-i} \), where \( D = \frac{d}{d x} + \frac{d^{2}}{d x^{2}} \) is superderivation, and it was also studied in [11].

Here we propose another supergeneralization of KP equation, whence, after reduction, there follows the bi-Hamiltonian s-KdV superequation (20).

The existence of an infinite number of motion integrals in involution is proved for this equation and the hierarchy of higher equations is constructed.

As it was noted in the paper [8] the KP equation can be correctly represented as the Hamiltonian system only in the space of periodic or almost periodic functions. The same statement is true for the proposed KP superequation.

1. Symbols Algebra

We consider the formal series of the type \( X = \sum_{i=n}^{\infty} X_{i} e^{t} \), where \( X_{i} \) are smooth functions on a circle admitting the value in the even sector of some Grassman algebra \( \mathfrak{g} \) (possibly, infinite). This means that the functions may involve the bilinear, fourlinear, etc. combinations of odd functions. The symbols algebra \( \mathfrak{g} \) is converted to the Lie algebra by the operation

\[ [X, Y] = X \circ Y - Y \circ X, \]

where the product \( X \circ Y \) is determined by the formula

\[ X \circ Y = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d^{i}}{d t^{i}} X_{i} e^{t} Y. \]

There are two subalgebras in the algebra \( \mathfrak{g} \), \( \mathfrak{g}_{-} \) - the algebra of differential operators and \( \mathfrak{g}_{+} \) - the algebra of formal integral operators. The elements \( 1 + X, X \in \mathfrak{g}_{-} \) form the group \( \mathfrak{g}_{-} \). The bilinear form in the algebra can be determined as

\[ (X, Y) = \int dx \, \text{Res}_{\Delta} (X \circ Y), \]

(4)

which allows one to identify \( \mathfrak{g}_{-}^{*} \) with \( \mathfrak{g}_{+} \).

Using the construction of the paper [9] we define the central extension of symbols algebra. Let \( \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{C} \otimes \mathfrak{C}^{\infty} (\mathfrak{S}) \) be the algebra of the function on a circle \( \mathfrak{S} \), \( \mathfrak{S} \in \mathfrak{S} \), taking the values in \( \mathfrak{g} \) and with the pointwise commutator. One can identify the \( \mathfrak{g}^{*} \) with \( \mathfrak{g} \) by means of the scalar product \( (X, Y) = \int dx \, \text{Res}_{\Delta} (X \circ Y) \). The formula

\[ \omega (X, Y) = (X, \frac{dY}{dx}), \]

(5)

sets the two-cocycle on \( \mathfrak{g} \). Let \( \mathfrak{g} \) be the central extension of \( \mathfrak{g} \) constructed by means of the cocycle (5). The coadjoint representation of \( \mathfrak{g} \) acts on the space \( \mathfrak{g}^{*} \otimes \mathfrak{g} \) by the formula

\[ ad^{*} A (L, c) = ([A, L] + c \frac{dA}{dx}, L), \]

(6)
The group coadjoint action that corresponds to (6) is determined by the relation
\[ Ad^*\varphi(L, c) = (\varphi(L) - \varphi(-1) + c \frac{d\varphi}{dy} \varphi^{-1}, c), \] (7)
where we consider only the case when \( \varphi \in \mathcal{G}_\mathcal{G} = C^\infty(\mathcal{B}, \mathcal{G}_\mathcal{G}) \).

Fixing \( c = 1 \) we describe the invariants of the action (7) on the subspaces \( \mathfrak{M} \subset \mathfrak{g}^* \) consisting of the operators such as
\[ \mathcal{L} = \mathcal{F}^N + u_{N-2} \mathcal{F}^{N-2} + \ldots + u_0 + u_{-1} \mathcal{F}^{-1} + \ldots \] (8)
Using the product (3) it is easy to prove the next

Lemma [8]. Any operator of the type (8) by means of the gauge transformation (7) is reduced to the form
\[ \mathcal{L}_0 = \mathcal{F}^N + \sum_{n=2}^{\infty} c_n \mathcal{F}^{-n}, \quad \varphi_x c_n = 0, \] (9)
and \( \varphi \) is uniquely fixed under the condition \( \varphi |_{x=0} = 1 \).

The functionals
\[ \mathcal{H}_n = \int dy \, c_n(y) \] (10)
form the involutory set of the integrals relative to Lie-Poisson bracket, which is induced by the action of algebra \( \mathfrak{g}_\mathcal{G} \) on the orbit of coadjoint action of the group \( \mathcal{G}_\mathcal{G} \).

The last statement is the corollary of the general theorem on the commutativity, proved in Ref. [10], or it can be proved independently, as it was made in paper [8].

2. KP Superequation

Let's use the above techniques to find the involutory integrals when
\[ \mathcal{L} = \mathcal{F}^2 + u + E_1 \mathcal{F}^{-1} E_2 \] (11)
and for the functions \( u(x, y), E_i(x, y) \) we set the conditions
\[ \int u(x, y) \, dx = 0, \quad \int E_1(x, y) E_2(x, y) \, dx = 0. \] (12)

The functions \( u(x, y), E_1(x, y) \) and \( E_2(x, y) \) are periodic on the variables \( x \) and \( y \) with the periods being equal 1, and \( u(x, y) \) takes the value in \( \mathcal{G}_\mathcal{G} \) - even sector of Grassman algebra \( \mathcal{G}_\mathcal{G} \), \( E_i(x, y) \) take the values in \( \mathcal{G}_\mathcal{G} \) - odd sector of the algebra \( \mathcal{G}_\mathcal{G} \).

Using Lemma we define the operator \( \Psi(x, y, \bar{y}) \) and the functions \( C_n(y) \) from the relation
\[ \Psi^{-1} \left( \frac{d}{dy} - L \right) \Psi = \frac{d}{dy} - L_0. \] (13)
For this we use the Rikhat substitution
\[ \Psi = \exp \left( \varphi(x, y, \bar{y}) \right), \] (14)
where the operator \( \varphi(x, y, \bar{y}) \) is represented in the form
\[ \varphi(x, y, \bar{y}) = \sum_{l=1}^{d} \varphi_l(x, y, \bar{y}) \mathcal{F}^{-l}. \]

Consecutively we obtain from the relation (13)
\[ C_0 = 2 \varphi_{2x} + u \]
\[ C_+ = 2 \varphi_{2y} + \varphi_1 y + \varphi_{1xx} + E_1 E_2 \]
\[ C_2 = 2 \varphi_{3x} - \varphi_2 y + \varphi_{2xx} + \varphi_{1x} - E_1 E_2 x \]
\[ C_3 = 2 \varphi_{3y} - \varphi_2 y + \varphi_{1xx} + 2 \varphi_{2xx} + E_1 E_2 x - E_1 E_2 \]
\[ C_3 = 2 \varphi_{4x} - \varphi_3 y + \varphi_{3xx} + 2 \varphi_{2xx} + \varphi_{1xx} - E_1 E_2 \]
\[ C_4 = 2 \varphi_{4y} - \varphi_3 y + \varphi_{3xx} + 2 \varphi_{2xx} + \varphi_{1xx} - E_1 E_2 \]
\[ \ldots \ldots \ldots \ldots \]
\[ C_n = 2 \varphi_{n+1x} + \varphi_{n+1} y + \varphi_{n+1xx} + \varphi_{1xx} - \ldots - E_1 E_2 \]

Using the fact that \( \varphi_x C_n = 0 \) we get from the condition of solvability of the n-th equation in (15) in periodic functions
and the condition $\psi|_{x=0} = 1$ fixes uniquely the choice of antiderivatives to find the functions $\psi_n(x, y)$.

The functions $c_n(y)$ are determined successively from (15) and having integrated them by the variable $y$ we obtain the functionals $H_n$, first five ones will be of the form

$$
H_0 = \int dx \, d\beta \, u(x, y) = 0, \\
H_1 = \int dx \, d\beta \, \varepsilon_4(x, y) \varepsilon_2(x, y) = 0, \\
H_2 = \int dx \, d\beta \left[ \frac{1}{4} u^2 - \varepsilon_4 \varepsilon_2 \right], \\
H_3 = \int dx \, d\beta \left[ \frac{1}{4} u \partial_x^3 u_y + \varepsilon_4 \varepsilon_2 \right], \\
H_4 = \int dx \, d\beta \left[ \frac{1}{8} u^3 - \frac{3}{16} (u_x)^2 + \frac{3}{16} (\partial_x u_y)^2 - \varepsilon_4 \varepsilon_2 ight].
$$

Using the operator of Hamiltonian structure $\Omega_4$

$$
\Omega_4 = \text{diag} (2\partial_x, -1, -1)
$$

and choosing $H_4$ as Hamiltonian we receive the next system of equations

$$
\begin{align*}
\dot{u}_x &= \frac{1}{16} u_{xxx} + \frac{1}{4} u_{xx} u_x + \frac{3}{8} (\partial_x u_y)^2 + \frac{3}{16} (\partial_x \partial_y u_y) + \frac{3}{4} \varepsilon_4 \varepsilon_2 \varepsilon_2, \\
\dot{\varepsilon}_2 &= \dot{\varepsilon}_4, \\
\dot{\varepsilon}_1 &= \frac{1}{4} u \varepsilon_2 + \frac{3}{4} \varepsilon_2 \varepsilon_4 \partial_x u_y - \frac{1}{4} \varepsilon_2 \partial_x u_y, \\
\dot{\varepsilon}_4 &= \frac{1}{4} u \varepsilon_2 + \frac{3}{4} \varepsilon_2 \varepsilon_4 \partial_x u_y - \frac{1}{4} \varepsilon_2 \partial_x u_y.
\end{align*}
$$

The appearance of such term as $\frac{1}{4} \int dx \, (\partial_x^2 u_y u_y)$ in (19) is observed for the usual KdV equation [8] and is connected with the conditions of coordination with (12).

It is easy to see, when the $L$ operator (11) is compared with $L$ operator for bi-Hamiltonian superequation $s$-KdV [3]

$$
\begin{align*}
\dot{u}_x &= \frac{1}{4} u_{xxx} + \frac{3}{2} u_{xx} u_x - \frac{3}{4} \varepsilon_{xxx}, \\
\dot{\varepsilon}_2 &= \dot{\varepsilon}_4, \\
\dot{\varepsilon}_1 &= \frac{1}{4} u \varepsilon_2 + \frac{3}{4} \varepsilon_2 \varepsilon_4 \partial_x u_y - \frac{1}{4} \varepsilon_2 \partial_x u_y,
\end{align*}
$$

how one can make a reduction from equations (19) to equations (20). One should take the next relations

$$
\frac{\partial u}{\partial y} = \frac{\partial \varepsilon_2}{\partial y} = \frac{\partial \varepsilon_4}{\partial y} = 0, \\
\varepsilon_2 = \varepsilon_4.
$$

In this connection the integrals with odd number vanish, as it was in purely bosonic case.

The Hamiltonians constructed by means of higher Hamiltonians $H_n$, $n > 4$ can be represented as equations of trivial curvature. Let $t_n$ be the proper time for every Hamiltonian $H_n$, then the hierarchy of higher equations can be written in the form

$$
\frac{\partial L}{\partial t_n} - \frac{\partial A_n}{\partial y} = [A_n, L],
$$

where $A_n = (\mathcal{L} \circ \mathcal{L}^{-1} \circ \mathcal{L}^{-1} \circ \mathcal{L}^{-1})$ and symbol $(\mathcal{B})^+$ means the differential part of the operator $\mathcal{B}$. For example, the operator $A_3 = (\mathcal{L} \circ \mathcal{L}^{-1} \circ \mathcal{L}^{-1})$, where $\mathcal{L}$ is determined by (15), has the form

$$
A_3 = \partial_x^3 + \frac{3}{2} u \partial_x + \frac{3}{4} (u_x + \partial_x u_y + 2 \partial_x \varepsilon_2),
$$

and equation (22) is equal to the system of equations (19). Under reduction (21) the operator $A_3$ is transformed to the $A$ operator from $L-A$ pair for bi-Hamiltonian superequation $s$-KdV [3].

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